A NEW INTERPRETATION OF INTERVAL-VALUED FUZZY INTERIOR IDEALS OF ORDERED SEMIGROUPS

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ABSTRACT *The concept of interval-valued fuzzy set is one of the most important and useful generalisation of Zadeh's fuzzy sets which is successfully applied by engineers and scientists in the field of Robotics, Control Theory and Computer Engineering. In this paper, we introduce new and useful generalisations of interval-valued interior ideals and interval-valued fuzzy left (right) ideals called interval-valued* $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$*-fuzzy interior ideals, interval-valued* $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$*-fuzzy left (right) ideals, interval-valued* $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$*-fuzzy interior ideals and interval-valued* $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$*-fuzzy left (right) ideals of ordered semigroups. These newly defined concepts are supported by suitable examples, and several fundamental results are investigated. It is shown that in regular and semisimple ordered semigroups, both the concepts of interval-valued* $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$*-fuzzy interior ideals and interval-valued* $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$*-fuzzy ideals coincide. Further, the relation between interval-valued fuzzy interior ideals and interval-valued fuzzy ideal of type* $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ *is provided.*

Keywords Interval-valued fuzzy interior ideals; Interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideals; interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy left (right) ideals; $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideals; $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy left (right) ideals.

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1. INTRODUCTION

Representation of knowledge in decision making process by means of intervals is more precise rather than by points. An interval-valued fuzzy subset [1,2] is a natural extension of fuzzy set theory [3] and more applicable in real world problems involving uncertainties. For the first time Biswas [4] used interval-valued fuzzy sets in algebraic structure and gave the notion of interval-valued fuzzy subgroups. In addition, Shabir and Khan [5] introduced interval-valued fuzzy left (right, two-sided, interior, bi-) ideals generated by an interval-valued fuzzy subset of ordered semigroups. Further, Khan *et al.* [6] initiated a new sort of interval-valued fuzzy bi-ideals known as interval-valued $(\in, \in \vee q)$-fuzzy bi-ideals of ordered semigroups. In addition, Khan et. al., [7,8] defined interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy generalized bi-ideal and interval-valued fuzzy ideals of type $(\in, \in \vee q_{\tilde{k}})$ of ordered semigroups and characterized ordered semigroups in terms of these notions. Moreover, Yin and Zhan [9] introduced $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$-fuzzy (implicative, positive implicative and fantastic) filters of *BL*-algebras and gave some interesting results. Further, Ma *et al.*, [10] initiated the concept of $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy ideals of *BL*-algebras and discussed several important results. In addition, Khan *et al.*, [11] gave more general forms of $(\in, \in \vee q)$-fuzzy interior ideals and $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy interior ideals of ordered semigroups and defined the concepts of fuzzy interior ideals and fuzzy left (resp. right) ideals of types $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ and $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ and characterised ordered semigroup by the properties of these new notions. Moreover, Khan *et al.*, [12] comprehensively discussed $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy generalized bi-ideals of ordered semigroups and provided several classifications of ordered semigroups in terms of $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy generalized bi-ideals. Moreover, Aktas and Çagman [13] introduced the concepts of fuzzy subring, fuzzy ideal and fuzzy ring homomorphism.

In this paper, we extend the work of [11,12] and introduced interval-valued fuzzy interior ideals (resp. ideals) of types $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ and $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$, where $\tilde{\gamma}, \tilde{\delta} \in D[0,1]$ such that $\tilde{\gamma} < \tilde{\delta}$. Moreover, a condition is provided that when both interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideals and interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy ideals will coincide.

2. PRELIMINARIES

In this section, we review some fundamental concepts that are necessary for this paper.

Throughout this paper S will denote an *ordered semigroup* unless otherwise stated.

For $A \subseteq S$, we denote $(A] := \{t \in S \mid t \le h \text{ for some } h \in A\}$. If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $A, B \subseteq S$, we denote, $AB := \{ab \mid a \in A, b \in B\}$. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-epmty subset A of S is called an *interior ideal* of S if (i) $A^2 \subseteq A$, (ii) $SAS \subseteq A$ and (iii) if $b \in S$ and $b \le a \in A$ then $b \in A$. A non-empty subset A of S is called *left* (resp. *right*) *ideal* of S if (i) $SA \subseteq A$ (resp. $AS \subseteq A$) and (ii) If $b \in S$ and $b \le a \in A$ then $b \in A$. A non-empty subset A of S is an *ideal* if it is both a left and a right ideal of S.

Obviously, every ideal of an ordered semigroup S is an interior ideal of S.

Now we recall some interval-valued fuzzy logic concepts.

2.1 INTERVAL-VALUED FUZZY SET[1]

For a non-empty set X a mapping $\tilde{F}: X \to D[0,1]$ is called an interval-valued fuzzy set of X, where $D[0,1]$ denotes the family of all closed subintervals of $[0,1]$, and $\tilde{F} = [F^-(x), F^+(x)]$ for all $x \in X$, where F^-, F^+ are fuzzy sets in X such that $0 \le F^-(x) \le F^+(x) \le 1$ for all $x \in X$ and $[F^-(x), F^+(x)]$ is the grade of membership of an element x to the set $\tilde{F}$.

2.2 INTERVAL-VALUED ORDERED FUZZY POINT

An interval-valued fuzzy subset $\tilde{F}$ of an ordered semigroup S of the form:

$$\tilde{F}(y) := \begin{cases} \tilde{t} \in D(0,1], & \text{if} \quad y \in (x], \\ [0,0], & \text{if} \quad y \notin (x], \end{cases}$$

is called an interval-valued fuzzy point with support x and value $\tilde{t}$ and is denoted by $x_{\tilde{t}}$.

If $\tilde{F}(x) \ge \tilde{t}$ (resp. $\tilde{F}(x) + \tilde{t} > \tilde{1}$) then we say that $x_{\tilde{t}}$ belongs to (resp. $x_{\tilde{t}}$ quasi-coincident with) a fuzzy set $\tilde{F}$, written as $x_{\tilde{t}} \in \tilde{F}$ (resp. $x_{\tilde{t}} \mathrm{q} \tilde{F}$) and write $x_{\tilde{t}} \in \vee \mathrm{q} \tilde{F}$ if $x_{\tilde{t}} \in \tilde{F}$ or $x_{\tilde{t}} \mathrm{q} \tilde{F}$.

2.3 LEVEL SET OF AN INTERVAL-VALUED FUZZY SET[1]

Let $\tilde{F}$ be an interval-valued fuzzy subset of X. Then, the crisp set $U(\tilde{F};\tilde{t}) = \{x \in X \mid \tilde{F}(x) \ge \tilde{t}\}$ for every $\tilde{0} < \tilde{t} \le \tilde{1}$ is called a level subset of $\tilde{F}$.

2.4 INTERVAL-VALUED FUZZY INTERIOR IDEAL[6]

An interval-valued fuzzy subset $\tilde{F}$ of an ordered semigroup S is called an interval-valued fuzzy interior ideal of S if the following two conditions hold for all $x, y, z \in S$:

(I_1) $\tilde{F}(xyz) \ge \tilde{F}(y)$,

(I_2) $x \le y \Rightarrow \tilde{F}(x) \ge \tilde{F}(y)$,

(I_3) $\tilde{F}(xy) \ge \mathrm{r}\min\{\tilde{F}(x), \tilde{F}(y)\}$.

2.5 INTERVAL-VALUED FUZZY LEFT (RIGHT) IDEAL[6]

An interval-valued fuzzy subset $\tilde{F}$ of an ordered semigroup S is called an interval-valued fuzzy left (resp. right) ideal of S if the following conditions hold for all $x, y \in S$:

(I_4) $x \le y \Rightarrow \tilde{F}(x) \ge \tilde{F}(y)$,

(I_5) $\tilde{F}(xy) \ge \tilde{F}(y)$ (resp. $\tilde{F}(xy) \ge \tilde{F}(x)$).

$\tilde{F}$ is called an interval-valued fuzzy ideal of S if it is both interval-valued fuzzy left and interval-valued fuzzy right ideal of S.

3. INTERVAL-VALUED $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}})$-FUZZY INTERIOR IDEALS

In this section we give another useful generalisation of interval-valued fuzzy interior ideals and interval-valued fuzzy left (right) ideals called interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}})$-fuzzy interior ideal and interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}})$-fuzzy left (right) ideals. In addition, we provide several characterisations of ordered semigroup in terms of these new concepts.

Throughout in this paper, let $\tilde{\gamma}, \tilde{\delta} \in D[0,1]$ be such that $\tilde{\gamma} < \tilde{\delta}$. For an interval-valued ordered fuzzy point $x_{\tilde{t}}$ and an interval-valued fuzzy subset $\tilde{F}$ of S, we say that

- $x_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ if $\tilde{F}(x) \ge \tilde{t} > \tilde{\gamma}$.
- $x_{\tilde{t}} \mathrm{q}_{\tilde{\delta}} \tilde{F}$ if $\tilde{F}(x) + \tilde{t} > 2\tilde{\delta}$.
- $x_{\tilde{t}} \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}} \tilde{F}$ if $x_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ or $x_{\tilde{t}} \mathrm{q}_{\tilde{\delta}} \tilde{F}$.
- $x_{\tilde{t}} \in_{\tilde{\gamma}} \wedge \mathrm{q}_{\tilde{\delta}} \tilde{F}$ if $x_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ and $x_{\tilde{t}} \mathrm{q}_{\tilde{\delta}} \tilde{F}$.
- $x_{\tilde{t}} \overline{\alpha} \tilde{F}$ if $x_{\tilde{t}} \alpha \tilde{F}$ does not hold for $\alpha \in \{\in_{\tilde{\gamma}}, \mathrm{q}_{\tilde{\delta}}, \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}}, \in_{\tilde{\gamma}} \wedge \mathrm{q}_{\tilde{\delta}}\}$.

3.1 DEFINITION

An interval-valued fuzzy subset $\tilde{F}$ of S is called an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}})$-fuzzy interior ideal of S if:

(a). $(\forall\, x \le y)\, (y_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F} \Rightarrow x_{\tilde{t}} \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}} \tilde{F})$,

(b). $x_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F},\ y_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F} \Rightarrow (xy)_{\mathrm{r}\min\{\tilde{t},\tilde{s}\}} \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}} \tilde{F}$,

(c). $a_{\tilde{t}} \in_{\tilde{\gamma}} F \Rightarrow (xay)_{\tilde{t}} \in_{\tilde{\gamma}} \vee \mathrm{q}_{\tilde{\delta}} \tilde{F}$, for all $x, a, y \in S$ and $\tilde{t}, \tilde{s} \in D(\tilde{\gamma}, 1]$.

3.2 EXAMPLE

Consider the ordered semigroup $S = \{0,1,2,3\}$ with the following multiplication table and order relation:

Table: 1

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(0,1)\}$$

Define an interval-valued fuzzy subset $\tilde{F} : S \to D[0,1]$ as follows:

$$\tilde{F}(x) = \begin{cases} [0.80, 0.90], & \text{if } x = 0, \\ [0.40, 0.50], & \text{if } x = 1, \\ [0.70, 0.80], & \text{if } x = 2, \\ [0.00, 0.00], & \text{if } x = 3. \end{cases}$$

Then $\tilde{F}$ is an interval valued $(\in_{[0.1,0.2]}, \in_{[0.1,0.2]} \vee q_{[0.6,0.7]})$-fuzzy interior ideal of S.

3.3 THEOREM

For an interval-valued fuzzy subset $\tilde{F}$ of S, the following conditions are equivalent;

(1). $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S.

(2). The following conditions hold for all $x, a, y \in S$:

(2.1). $(\forall\ x \leq y)\ (\text{r}\max\{\tilde{F}(x), \tilde{\gamma}\} \geq \text{r}\min\{\tilde{F}(y), \tilde{\delta}\})$,

(2.2). $\text{r}\max\{\tilde{F}(xy), \tilde{\gamma}\} \geq \text{r}\min\{\tilde{F}(x), \tilde{F}(y), \tilde{\delta}\}$,

(2.3). $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} \geq \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$.

Proof. Let $\tilde{F}$ be an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S. If there exist $a, b \in S$ such that $x \leq y$ and $\text{r}\max\{\tilde{F}(a), \tilde{\gamma}\} < \text{r}\min\{\tilde{F}(b), \tilde{\delta}\}$, then $\text{r}\max\{\tilde{F}(a), \tilde{\gamma}\} < \tilde{t} \leq \text{r}\min\{\tilde{F}(b), \tilde{\delta}\}$ for some $\tilde{t} \in D(\tilde{\gamma}, 1]$. It follows that $\tilde{F}(b) \geq \tilde{t} > \tilde{\gamma}$ but $\tilde{F}(a) < \tilde{t}$ and $\tilde{F}(a) + \tilde{t} < 2\tilde{t} \leq 2\tilde{\delta}$, it follows that $a_{\tilde{t}} \overline{\in}_{\tilde{\gamma}} \tilde{F}$ and $a_{\tilde{t}} \overline{q}_{\tilde{\delta}} \tilde{F}$ where, $b_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$, a contradiction and hence (2.1) is acceptable for all $x, y \in S$.

If $\text{r}\max\{\tilde{F}(ab), \tilde{\gamma}\} < \text{r}\min\{\tilde{F}(a), \tilde{F}(b), \tilde{\delta}\}$ for some $a, b \in S$, then there exists $\tilde{s} \in D(\tilde{\gamma}, 1]$ such that $\text{r}\max\{\tilde{F}(ab), \tilde{\gamma}\} < \tilde{s} \leq \text{r}\min\{\tilde{F}(a), \tilde{F}(b), \tilde{\delta}\}$. This implies $a_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F}$, $b_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F}$ but $(ab)_{\tilde{s}} \overline{\in}_{\tilde{\gamma}} \tilde{F}$ and $(ab)_{\tilde{s}} \overline{q}_{\tilde{\gamma}} \tilde{F}$, again a contradiction and therefore we accept that (2.2) is valid for all $x, y \in S$.

Suppose that there exist $x, a, y \in S$ such that $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} < \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$. Then $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} < \tilde{t} \leq \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$ for some $\tilde{s} \in D(\tilde{\gamma}, 1]$, it follows that $a_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F}$ but $(xay)_{\tilde{s}} \overline{\in}_{\tilde{\gamma}} \tilde{F}$ and $(xay)_{\tilde{s}} \overline{q}_{\tilde{\gamma}} \tilde{F}$, contradicting Condition (c) of Definition (3.1). Hence (2.3) is true for all $x, a, y \in S$.

Conversely, we assume that Conditions (2.1), (2.2) and (2.3) are satisfied for all $x, a, y \in S$.

Let there exist $x, y \in S$ with $x \leq y$ and $\tilde{t} \in D(\tilde{\gamma}, 1]$ such that $y_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ but $x_{\tilde{t}} \overline{\in}_{\tilde{\gamma}} \tilde{F}$ and $x_{\tilde{t}} \overline{q}_{\tilde{\delta}} \tilde{F}$. Then $\tilde{F}(y) \geq \tilde{t} > \tilde{\gamma}$, $\tilde{F}(x) < \tilde{t}$ and $\tilde{F}(x) + \tilde{t} < 2\tilde{\delta}$, showing that $\tilde{F}(x) < \tilde{\delta}$. Hence $\text{r}\max\{\tilde{F}(x), \tilde{\gamma}\} < \text{r}\min\{\tilde{t}, \tilde{\delta}\} \leq \text{r}\min\{\tilde{F}(y), \tilde{\delta}\}$, this contradicts (2.1). Hence (a) is valid.

If $a_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F}$, $b_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ for some $a, b \in S$ and $\tilde{s}, \tilde{t} \in D(\tilde{\gamma}, 1]$ such that $(ab)_{\text{r}\min\{\tilde{s}, \tilde{t}\}} \overline{\in}_{\tilde{\gamma}} \tilde{F}$ and $(ab)_{\text{r}\min\{\tilde{s}, \tilde{t}\}} \overline{q}_{\tilde{\delta}} \tilde{F}$, then $\tilde{F}(a) \geq \tilde{s} > \tilde{\gamma}$, $\tilde{F}(b) \geq \tilde{t} > \tilde{\gamma}$ and $\tilde{F}(ab) < \text{r}\min\{\tilde{t}, \tilde{s}\}$, $\tilde{F}(ab) + \text{r}\min\{\tilde{t}, \tilde{s}\} < 2\tilde{\delta}$. This shows that $\tilde{F}(ab) < 2\tilde{\delta}$ and therefore $\text{r}\max\{\tilde{F}(xy), \tilde{\gamma}\} < \text{r}\min\{\tilde{t}, \tilde{s}\} \leq \text{r}\min\{\tilde{F}(x), \tilde{F}(y), \tilde{\delta}\}$, which contradicts (2.2). Hence (b) is valid.

If there exist $x, a, y \in S$ and $\tilde{t} \in D(\tilde{\gamma}, 1]$ such that $a_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F}$ but $(xay)_{\tilde{t}} \overline{\in}_{\tilde{\gamma}} \tilde{F}$ and $(xay)_{\tilde{t}} \overline{q}_{\tilde{\delta}} \tilde{F}$, then $\tilde{F}(a) \geq \tilde{t} > \tilde{\gamma}$, $\tilde{F}(xay) < \tilde{t}$ and $\tilde{F}(xay) + \tilde{t} < 2\tilde{\delta}$, showing that $\tilde{F}(xay) < 2\tilde{\delta}$. Hence $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} < \text{r}\min\{\tilde{t}, \tilde{\delta}\} \leq \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$, contradicts (2.3) and therefore (c) is valid. Consequently, $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S.

3.4 THEOREM

For an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal $\tilde{F}$ of S, the set $\tilde{F}_{\tilde{\gamma}} = \{x \in S \mid \tilde{F}(x) > \tilde{\gamma}\}$ is an interior ideal of S if $2\tilde{\delta} = \tilde{1} + \tilde{\gamma}$.

Proof. Let $\tilde{F}$ be an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S and $a, b \in \tilde{F}_{\tilde{\gamma}}$, then $\tilde{F}(a) > \tilde{\gamma}$ and $\tilde{F}(b) > \tilde{\gamma}$. Hence by (b) of Definition (3.1) $(ab)_{\text{r}\min\{\tilde{F}(a), \tilde{F}(b)\}} \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}} \tilde{F}$ i.e., $\tilde{F}(ab) \geq \text{r}\min\{\tilde{F}(a), \tilde{F}(b)\} > \tilde{\gamma}$ or $\tilde{F}(ab) > 2\tilde{\delta} - \text{r}\min\{\tilde{F}(a), \tilde{F}(b)\} \geq 2\tilde{\delta} - \tilde{1} = \tilde{\gamma}$. This shows that $ab \in \tilde{F}_{\tilde{\gamma}}$.

If there exist $x,a,y \in S$ such that $a \in \tilde{F}_{\tilde{\gamma}}$, then $\tilde{F}(a) > \tilde{\gamma}$ and by (c) of Definition (3.1) $(xay)_{\tilde{F}(a)} \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}} \tilde{F}$ i.e., $\tilde{F}(xay) \geq \tilde{F}(a) > \tilde{\gamma}$ or $\tilde{F}(xay) > 2\tilde{\delta} - \tilde{F}(a) \geq 2\tilde{\delta} - \tilde{1} = \tilde{\gamma}$, showing that $xay \in \tilde{F}_{\tilde{\gamma}}$. Hence $\tilde{F}_{\tilde{\gamma}}$ is an interior ideal of S.

Directly by Theorem (3.4) we have the following two corollaries.

3.5 COROLLARY

Let F be $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy interior ideal of S and $2\delta = 1 + \gamma$, then the set $F_{\gamma} = \{x \in S \mid F(x) > \gamma\}$ is an interior ideal of S.

3.6 COROLLARY

The set $F_0 = \{x \in S \mid F(x) > 0\}$ is an interior ideal of S $(\in, \in \vee q)$-fuzzy interior ideal F of S.

3.7 PROPOSITION

If $\{\tilde{F}_i\}_{i \in I} \neq \phi$ is a collection of interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S, then $\bigcap_{i \in I} \tilde{F}_i$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S.

Proof. Let $\tilde{F}_i$ be an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S for all $i \in I$ and $a, b \in S$ with $a \leq b$. Consider

$$\begin{aligned} \mathrm{r}\max\{\bigcap_{i\in I} \tilde{F}_i(a), \tilde{\gamma}\} &= \bigwedge_{i\in I}\{\mathrm{r}\max\{\tilde{F}_i(a), \tilde{\gamma}\}\} \\ &\geq \bigwedge_{i\in I}\{\mathrm{r}\min\{\tilde{F}_i(b), \tilde{\delta}\}\} \\ &\quad (\text{By Theorem } 3.3(2.1)) \\ &= \mathrm{r}\min\{\bigwedge_{i\in I} \tilde{F}_i(b), \tilde{\delta}\} \\ &= \mathrm{r}\min\{(\bigcap_{i\in I} \tilde{F}_i)(b), \tilde{\delta}\}. \end{aligned}$$

Next we take $a, b \in S$ and consider

$$\begin{aligned} \mathrm{r}\max\{\bigcap_{i\in I} \tilde{F}_i(ab), \tilde{\gamma}\} &= \bigwedge_{i\in I}\{\mathrm{r}\max\{\tilde{F}_i(ab), \tilde{\gamma}\}\} \\ &\geq \bigwedge_{i\in I}\{\mathrm{r}\min\{\tilde{F}_i(a), \tilde{F}_i(b), \tilde{\delta}\}\} \\ &\quad (\text{By THeorem } 3.3(2.2)) \\ &= \mathrm{r}\min\{\bigwedge_{i\in I} \tilde{F}_i(a), \bigwedge_{i\in I} \tilde{F}_i(b), \tilde{\delta}\} \\ &= \mathrm{r}\min\{(\bigcap_{i\in I} \tilde{F}_i)(a), (\bigcap_{i\in I} \tilde{F}_i)(b), \tilde{\delta}\}. \end{aligned}$$

Finally, if $x, a, y \in S$, then

$$\begin{aligned} \mathrm{r}\max\{\bigcap_{i\in I} \tilde{F}_i(xay), \tilde{\gamma}\} &= \bigwedge_{i\in I}\{\mathrm{r}\max\{\tilde{F}_i(xay), \tilde{\gamma}\}\} \\ &\geq \bigwedge_{i\in I}\{\mathrm{r}\min\{\tilde{F}_i(a), \tilde{\delta}\}\} \\ &\quad (\text{By THeorem } 3.3(2.3)) \\ &= \mathrm{r}\min\{\bigwedge_{i\in I} \tilde{F}_i(a), \tilde{\delta}\} \\ &= \mathrm{r}\min\{(\bigcap_{i\in I} \tilde{F}_i)(a), \tilde{\delta}\}. \end{aligned}$$

Consequently by Theorem 3.3 $\bigcap_{i \in I} \tilde{F}_i$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S.

Now it is natural to investigate that $\bigcup_{i \in I} \tilde{F}_i$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal of S or not for any non-empty collection $\{\tilde{F}_i\}_{i \in I}$ of interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideals of S. In this regard we constructed the following example to show that $\bigcup_{i \in I} \tilde{F}_i$ is not an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideal in general.

3.8 EXAMPLE

We consider the ordered semigroup $S = \{a, b, c, d\}$ defined by the following multiplication table and order relations.

Table: 2

.	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

$$\leq := \{(a,a),(b,b),(c,c),(d,d),(a,d)\}.$$

Define two interval-valued fuzzy subsets $\tilde{F}_1$ and $\tilde{F}_2$ as

$$\tilde{F}_1(x) = \begin{cases} [0.40, 0.50], & \text{if} \quad x \in \{a, b\}, \\ [0.00, 0.00], & \text{if} \quad x \in \{c, d\}, \end{cases}$$

and

$$\tilde{F}_2(x) = \begin{cases} [0.40, 0.50], & \text{if} \quad x \in \{a, c\}, \\ [0.00, 0.00], & \text{if} \quad x \in \{b, d\}. \end{cases}$$

Then $\tilde{F}_1$ and $\tilde{F}_2$ are interval-valued $(\in_{[0.2,0.3]}, \in_{[0.2,0.3]} \vee q_{[0.3,0.4]})$-fuzzy interior ideals of S. But $\tilde{F}_1 \cup \tilde{F}_2$ is not an interval-valued $(\in_{[0.2,0.3]}, \in_{[0.2,0.3]} \vee q_{[0.3,0.4]})$-fuzzy interior ideals of S. Since,

$$\begin{aligned} & \mathrm{r}\max\{(\tilde{F}_1 \cup \tilde{F}_2)(bc), \tilde{\gamma} = [0.20, 0.30]\} \\ &= \mathrm{r}\max\{(\tilde{F}_1 \cup \tilde{F}_2)(d), \tilde{\gamma} = [0.20, 0.30]\} \\ &= \mathrm{r}\max\left\{ \begin{array}{l} \mathrm{r}\max\{\tilde{F}_1(d) = [0,0], \tilde{F}_2(d) = [0,0], \\ \tilde{\gamma} = [0.20, 0.30] \end{array} \right\} \\ &= [0.20, 0.30] \end{aligned}$$

and

$$\begin{aligned} & \mathrm{r}\min\{(\tilde{F}_1 \cup \tilde{F}_2)(b), (\tilde{F}_1 \cup \tilde{F}_2)(c), \tilde{\delta} = [0.30, 0.40]\} \\ &= \mathrm{r}\min\left\{ \begin{array}{l} \mathrm{r}\max\{\tilde{F}_1(b) = [0.40, 0.50], \tilde{F}_2(b) = [0,0]\}, \\ \mathrm{r}\max\{\tilde{F}_1(c) = [0,0], \tilde{F}_2(c) = [0.40, 0.50]\}, \\ \tilde{\delta} = [0.30, 0.40] \end{array} \right\} \\ &= \mathrm{r}\min\{[0.40, 0.50], [0.40, 0.50], [0.30, 0.40]\} \\ &= [0.30, 0.40]. \end{aligned}$$

Hence

$\text{r}\max\{(\tilde{F}_1 \cup \tilde{F}_2)(bc), \tilde{\gamma} = [0.20, 0.30]\}$
$= [0.20, 0.30]$
$< [0.30, 0.40]$
$= \text{r}\min\{(\tilde{F}_1 \cup \tilde{F}_2)(b), (\tilde{F}_1 \cup \tilde{F}_2)(c), \tilde{\delta} = [0.30, 0.40]\}.$

3.9 DEFINITION

Let $\tilde{F}$ be an interval-valued fuzzy subset of S. Then $\tilde{F}$ is called an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}})$-fuzzy interior ideal of S if for all $x, a, y \in S$ and $\tilde{s}, \tilde{t} \in D(\tilde{\gamma}, 1]$, the following hold:

(d). $(\forall\ x \le y)\ (y_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F} \to x_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F})$,

(e). $x_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F},\ y_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F} \to (xy)_{\text{r}\min\{\tilde{s},\tilde{t}\}} \in_{\tilde{\gamma}} \tilde{F}$,

(f). $a_{\tilde{t}} \in_{\tilde{\gamma}} F \to (xay)_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$.

3.10 THEOREM

Every interval-valued fuzzy interior ideal of an ordered semigroup S is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}})$-fuzzy interior ideal of S.

Proof. Let $\tilde{F}$ be an interval-valued fuzzy interior ideal of S and $a, b \in S$ with $a \le b$ such that $b_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$. Then $F(\text{b}) \ge \tilde{t} > \tilde{\gamma}$ and by Definition 2.4 (I_2) $\tilde{F}(a) \ge \tilde{F}(b) \ge \tilde{t} > \tilde{\gamma}$, follows that $a_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$.

If there exist $a, b \in S$ and $\tilde{s}, \tilde{t} \in D(\tilde{\gamma}, 1]$ such that $a_{\tilde{s}} \in_{\tilde{\gamma}} \tilde{F}$ and $b_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$, then $F(a) \ge \tilde{s} > \tilde{\gamma}$ and $F(b) \ge \tilde{t} > \tilde{\gamma}$. By Definition 2.4 (I_3) $\tilde{F}(ab) \ge \text{r}\min\{\tilde{F}(a), \tilde{F}(b)\} \ge \text{r}\min\{\tilde{s}, \tilde{t}\} > \tilde{\gamma}$, in which it follows that $(ab)_{\text{r}\min\{\tilde{s},\tilde{t}\}} \in_{\tilde{\gamma}} \tilde{F}$.

Finally, if there exist $x, a, y \in S$ and $\tilde{t} \in D(\tilde{\gamma}, 1]$ such that $a_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$, then $F(a) \ge \tilde{t} > \tilde{\gamma}$ and by Definition 2.4 (I_1) $\tilde{F}(xay) \ge \tilde{F}(a) \ge \tilde{t} > \tilde{\gamma}$ i.e., $(xay)_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$. Hence $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}})$-fuzzy interior ideal of S.

3.11 COROLLARY

Every interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}})$-fuzzy interior ideal of S is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy interior ideal of S

From Theorem (3.10) and Corollary (3.11) we have the following corollary.

3.12 COROLLARY

Each interval-valued fuzzy interior ideal of S is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy interior ideal of S.

To link interval-valued fuzzy interior ideal and interval-valued fuzzy ideal of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$, first we define interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy left (resp. right) ideal in the following lines.

3.13 DEFINITION

An interval-valued fuzzy subset $\tilde{F}$ of S is called an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy left (resp. right) ideal of S if the following conditions hold:

(g). $(\forall\ x \le y)\ (y_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F} \to x_{\tilde{t}} \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}} \tilde{F})$,

(h). $y_{\tilde{t}} \in_{\tilde{\gamma}} F \to (xy)_{\tilde{t}} \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}} \tilde{F}$ (resp. $(yx)_{\tilde{t}} \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}} \tilde{F}$), for all $x, y \in S$ and $\tilde{t} \in D(\tilde{\gamma}, 1]$.

An interval-valued fuzzy subset $\tilde{F}$ of S is called an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy ideal of S if it is both interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy left and right ideal of S.

3.14 THEOREM

Let $\tilde{F}$ be an interval-valued fuzzy subset of S. Then $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy left (resp. right) ideal of S if and only if the following conditions hold for all $x, y \in S$:

(i) $(\forall\ x \le y)$ $(\text{r}\max\{\tilde{F}(x), \tilde{\gamma}\} \ge \text{r}\min\{\tilde{F}(y), \tilde{\delta}\})$,

(ii) $\text{r}\max\{\tilde{F}(xy), \tilde{\gamma}\} \ge \text{r}\min\{\tilde{F}(y), \tilde{\delta}\}$ (resp. $\text{r}\min\{\tilde{F}(x), \tilde{\delta}\}$)

.

Proof. It is straightforward and omitted.

3.15 Theorem

Every interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy ideal of S is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy interior ideal of S.

Proof. Let $\tilde{F}$ be an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy ideal of S. If $a, b \in S$, then by Theorem 3.14 (ii)

$$\begin{aligned}\text{r}\max\{\tilde{F}(ab), \tilde{\gamma}\} &\ge \text{r}\min\{\tilde{F}(b), \tilde{\delta}\}\\ &\ge \text{r}\min\{\tilde{F}(b), \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}, \tilde{\delta}\}\\ &= \text{r}\min\{\tilde{F}(a), \tilde{F}(b), \tilde{\delta}\}.\end{aligned}$$

Therefore $\text{r}\max\{\tilde{F}(xy), \tilde{\gamma}\} \ge \text{r}\min\{\tilde{F}(x), \tilde{F}(y), \tilde{\delta}\}$ for all $x, y \in S$.

If $x, a, y \in S$ such that $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} < \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$, then $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} < \tilde{t} \le \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$ for some $\tilde{t} \in D(\tilde{\gamma}, 1]$, showing that $a_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$. By hypothesis $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy ideal, therefore $(x(ay))_{\tilde{t}} \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}} \tilde{F}$, but here we observe that $\tilde{F}(xay) < \tilde{t}$ and $\tilde{F}(xay) + \tilde{t} < 2\tilde{t} \le 2\tilde{\delta}$. It follows that $(xay)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ and $(xay)_{\tilde{t}}\ \overline{\text{q}}_{\tilde{\delta}} \tilde{F}$, a contradiction and hence $\text{r}\max\{\tilde{F}(xay), \tilde{\gamma}\} \ge \text{r}\min\{\tilde{F}(a), \tilde{\delta}\}$ for all $x, a, y \in S$. Consequently, $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee \text{q}_{\tilde{\delta}})$-fuzzy interior ideal of S.

The following example shows that the converse of Theorem 3.15 is not true in general.

3.16 EXAMPLE

Consider the ordered semigroup $S=\{0,1,2,3\}$ and the interval-valued fuzzy subset $\tilde{F}:S\to[0,1]$ as defined in Example (3.2). Then $\tilde{F}$ is an interval-valued $(\in_{[0.1,0.2]},\in_{[0.1,0.2]}\vee q_{[0.6,0.7]})$ -fuzzy interior ideal of S. Since $\tilde{F}(abc)=\tilde{F}(0)=[0.8,0.9]$, then

$$\begin{aligned}&\text{r max}\{\tilde{F}(abc)=[0.8,0.9],\tilde{\gamma}=[0.1,0.2]\}\\&=[0.8,0.9]\\&\geq \text{r min}\{\tilde{F}(b),\tilde{\delta}=[0.6,0.7]\}.\end{aligned}$$

If $ab=0$, then $\tilde{F}(ab)=\tilde{F}(0)=[0.8,0.9]$, and therefore

$$\begin{aligned}&\text{r max}\{\tilde{F}(ab)=[0.8,0.9],\tilde{\gamma}=[0.1,0.2]\}\\&=[0.8,0.9]\\&\geq \text{r min}\{\tilde{F}(b),\tilde{F}(b),\tilde{\delta}=[0.6,0.7]\}.\end{aligned}$$

On the other hand if $ab=1$, then $\tilde{F}(ab)=\tilde{F}(1)=[0.4,0.5]$ and so

$$\begin{aligned}&\text{r max}\{\tilde{F}(ab)=[0.4,0.5],\tilde{\gamma}=[0.1,0.2]\}\\&=[0.4,0.5]\\&>[0,0]\\&\geq \text{r min}\{\tilde{F}(b),\tilde{F}(b),\tilde{\delta}=[0.6,0.7]\}.\end{aligned}$$

If $ab=2$, then $\tilde{F}(ab)=\tilde{F}(2)=[0.7,0.8]$ and thus

$$\begin{aligned}&\text{r max}\{\tilde{F}(ab)=[0.7,0.8],\tilde{\gamma}=[0.1,0.2]\}\\&=[0.7,0.8]\\&>[0,0]\\&\geq \text{r min}\{\tilde{F}(b),\tilde{F}(b),\tilde{\delta}=[0.6,0.7]\}.\end{aligned}$$

Lastly, for $0\leq 1$, we can see that

$$\begin{aligned}&\text{r max}\{\tilde{F}(0)=[0.8,0.9],\tilde{\gamma}=[0.1,0.2]\}\\&=[0.8,0.9]\\&>\text{r min}\{\tilde{F}(1)=[0.4,0.5],\tilde{\delta}=[0.6,0.7]\}.\end{aligned}$$

The above discussion shows that $\tilde{F}$ is an interval-valued $(\in_{[0.1,0.2]},\in_{[0.1,0.2]}\vee q_{[0.6,0.7]})$ -fuzzy interior ideal of S.

However, $\tilde{F}$ is not an interval-valued $(\in_{[0.1,0.2]},\in_{[0.1,0.2]}\vee q_{[0.6,0.7]})$ -fuzzy ideal of S, since if $a=2$ and $b=3$, then $\tilde{F}(ab)=\tilde{F}(1)=[0.4,0.5]$ and thus

$$\begin{aligned}&\text{r max}\{\tilde{F}((2).(3))=[0.4,0.5],\tilde{\gamma}=[0.1,0.2]\}\\&=[0.4,0.5]\\&<[0.6,0.7]\\&=\text{r min}\{\tilde{F}(2)=[0.7,0.8],\tilde{\delta}=[0.6,0.7]\}.\end{aligned}$$

It follows that $\tilde{F}$ is not an interval-valued $(\in_{[0.1,0.2]},\in_{[0.1,0.2]}\vee q_{[0.6,0.7]})$ -fuzzy ideal of S.

In the following result we show that every interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy interior ideal of a regular ordered semigroup is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy ideal.

3.17 THEOREM

If $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy interior ideal of a regular ordered semigroup S', then $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy ideal of S'.

Proof. If $a,b\in S'$, then there exists $x\in S'$ such that $a\leq axa$. Since $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy interior ideal of S', therefore

$$\begin{aligned}\text{r max}\{\tilde{F}(ab),\tilde{\gamma}\}&\geq \text{r min}\{\tilde{F}(axa)(b),\tilde{\delta}\}\\&=\text{r min}\{\tilde{F}(ax)a(b),\tilde{\delta}\}\\&\geq \text{r min}\{\tilde{F}(a),\tilde{\delta}\}.\end{aligned}$$

Similarly, we can prove that

$$\text{r max}\{\tilde{F}(ab),\tilde{\gamma}\}\geq \text{r min}\{\tilde{F}(b),\tilde{\delta}\}.$$

It follows that $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy ideal of S'.

3.18 PROPOSITION

Every interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy interior ideal of a semisimple ordered semigroup S'' is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy ideal of S''.

Proof. Let $\tilde{F}$ be an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy interior ideal of S''. If $a,b\in S''$, then there exists $x,y,z\in S''$ such that $a\leq xayaz$. Therefore

$$\begin{aligned}\text{r max}\{\tilde{F}(ab),\tilde{\gamma}\}&\geq \text{r min}\{\tilde{F}(xayaz)(b),\tilde{\delta}\}\\&=\text{r min}\{\tilde{F}(xay)a(zb),\tilde{\delta}\}\\&\geq \text{r min}\{\tilde{F}(a),\tilde{\delta}\}.\end{aligned}$$

Similarly we can prove that

$$\text{r max}\{\tilde{F}(ab),\tilde{\gamma}\}\geq \text{r min}\{\tilde{F}(b),\tilde{\delta}\}$$

for all $a,b\in S''$. It follows that $\tilde{F}$ is an interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy ideal of S''.

From Theorem (3.15), Theorem (3.17) and Proposition (3.18) we have the following corollary.

3.19 COROLLARY

Interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy ideal and interval-valued $(\in_{\tilde{\gamma}},\in_{\tilde{\gamma}}\vee q_{\tilde{\delta}})$ -fuzzy interior ideal coincide in case of regular ordered semigroup and semisimple ordered semigroup.

4. INTERVAL-VALUED $(\overline{\in}_{\tilde{\gamma}},\overline{\in}_{\tilde{\gamma}}\vee\overline{q}_{\tilde{\delta}})$ -FUZZY INTERIOR IDEALS

In this section, we introduce interval-valued $(\overline{\in}_{\tilde{\gamma}},\overline{\in}_{\tilde{\gamma}}\vee\overline{q}_{\tilde{\delta}})$ -fuzzy interior ideals and interval-valued $(\overline{\in}_{\tilde{\gamma}},\overline{\in}_{\tilde{\gamma}}\vee\overline{q}_{\tilde{\delta}})$ -fuzzy left (right) ideals of ordered semigroup and characterise ordered semigroups by the properties of these newly defined interval-valued fuzzy ideals.

4.1 DEFINITION

An interval-valued fuzzy subset $\tilde{F}$ of S is called interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideal of S if for all $x, a, y \in S$ and $\tilde{t} \in D(\tilde{\delta}, 1]$ the following conditions are satisfied:

(i). $(\forall\ x \le y)\ (x_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F} \Rightarrow y_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F})$,

(j). $(xy)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F} \Rightarrow x_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$ or $y_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$,

(k). $(xay)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F} \Rightarrow a_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$.

4.2 EXAMPLE

Consider the ordered semigroup $S = \{0,1,2,3\}$ of Example (3.2) and define an interval-valued fuzzy subset $\tilde{F} : S \to [0,1]$ as follows:

$$\tilde{F}(x) = \begin{cases} [0.20, 0.30], & \text{if } x = 0, \\ [0.10, 0.20], & \text{if } x = 1, \\ [0.30, 0.40], & \text{if } x = 2, \\ [0.50, 0.60], & \text{if } x = 3. \end{cases}$$

Then $\tilde{F}$ is an interval-valued $(\overline{\in}_{[0.1,0.2]}, \overline{\in}_{[0.1,0.2]} \vee \overline{q}_{[0.6,0.7]})$-fuzzy interior-ideal of S.

4.3 THEOREM

A fuzzy subset $\tilde{F}$ of S is an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideal of S if and only if the following conditions hold for all $x, a, y \in S$:

(l). $(\forall\ x \le y)\ (\mathrm{r}\max\{\tilde{F}(x), \tilde{\delta}\} \ge \tilde{F}(y))$,

(m). $\mathrm{r}\max\{\tilde{F}(xy), \tilde{\delta}\} \ge \mathrm{r}\min\{\tilde{F}(x), \tilde{F}(y)\}$,

(n). $\mathrm{r}\max\{\tilde{F}(xay), \tilde{\delta}\} \ge \tilde{F}(a)$.

Proof. Let $\tilde{F}$ be an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideal of S and $a, b \in S$ with $a \le b$ such that $\mathrm{r}\max\{\tilde{F}(a), \tilde{\delta}\} < \tilde{F}(b)$. Then for some $\tilde{t} \in D(\tilde{\delta}, 1]$ we have $\mathrm{r}\max\{\tilde{F}(a), \tilde{\delta}\} < \tilde{t} \le \tilde{F}(b)$, follows that $a_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ but $b_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ and $b_{\tilde{t}} q_{\tilde{\delta}} \tilde{F}$, contradicting condition (i) of Definition (4.1). Hence Condition (l) is valid for all $x, y \in S$ with $x \le y$.

If there exist $a, b \in S$ such that $\mathrm{r}\max\{\tilde{F}(ab), \tilde{\delta}\} < \tilde{s} \le \mathrm{r}\min\{\tilde{F}(a), \tilde{F}(b)\}$ for some $\tilde{s} \in D(\tilde{\delta}, 1]$, then $(ab)_{\tilde{s}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$. By condition (j) of Definition (4.1) $a_{\tilde{s}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$ or $b_{\tilde{s}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$ that is $\tilde{F}(a) < \tilde{s}$ or $\tilde{F}(a) + \tilde{s} < 2\tilde{\delta}$ or $\tilde{F}(b) < \tilde{s}$ or $\tilde{F}(b) + \tilde{s} < 2\tilde{\delta}$, a contradiction. Hence $\mathrm{r}\max\{\tilde{F}(xy), \tilde{\delta}\} \ge \mathrm{r}\min\{\tilde{F}(x), \tilde{F}(y)\}$ for all $x, y \in S$.

Let $\mathrm{r}\max\{\tilde{F}(axb), \tilde{\delta}\} < \tilde{F}(x)$ for some $a, b, x \in S$. Then there exists $\tilde{t} \in D(\tilde{\delta}, 1]$ such that $\mathrm{r}\max\{\tilde{F}(axb), \tilde{\delta}\} < \tilde{t} \le \tilde{F}(x)$. This implies $(axb)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ but $x_{\tilde{t}} \in_{\tilde{\gamma}} \tilde{F}$ and $x_{\tilde{t}} q_{\tilde{\delta}} \tilde{F}$, a contradiction. Hence $\mathrm{r}\max\{\tilde{F}(xay), \tilde{\delta}\} \ge \tilde{F}(a)$ for all $x, a, y \in S$.

Conversely, assume that all the three conditions (l), (m) and (n) are satisfied by $\tilde{F}$ for all $x, a, y \in S$. If $a, b \in S$ with $a \le b$ such that $a_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$, then $\tilde{F}(a) < \tilde{t}$ and by (l)

$$\begin{aligned} \tilde{F}(b) &\le \mathrm{r}\max\{\tilde{F}(a), \tilde{\delta}\} \\ &< \mathrm{r}\max\{\tilde{t}, \tilde{\delta}\} \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \ge \tilde{\delta}, \\ \tilde{\delta}, & \text{if } \tilde{t} < \tilde{\delta}, \end{cases} \end{aligned}$$

in which it follows that $b_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$.

If $(xy)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ for $x, y \in S$, then $\tilde{F}(xy) < \tilde{t}$ and by (m)

$$\begin{aligned} \mathrm{r}\min\{\tilde{F}(x), \tilde{F}(y)\} &\le \mathrm{r}\max\{\tilde{F}(xy), \tilde{\delta}\} \\ &< \mathrm{r}\max\{\tilde{t}, \tilde{\delta}\} \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \ge \tilde{\delta}, \\ \tilde{\delta}, & \text{if } \tilde{t} < \tilde{\delta}. \end{cases} \end{aligned}$$

It follows that $x_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$ or $y_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$.

Finally, let $x, a, y \in S$ and $(xay)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$. Then $\tilde{F}(xay) < \tilde{t}$ and by (n)

$$\begin{aligned} \tilde{F}(a) &\le \mathrm{r}\max\{\tilde{F}(xay), \tilde{\delta}\} \\ &< \mathrm{r}\max\{\tilde{t}, \tilde{\delta}\} \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \ge \tilde{\delta}, \\ \tilde{\delta}, & \text{if } \tilde{t} < \tilde{\delta}. \end{cases} \end{aligned}$$

This shows that $a_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$. Hence $\tilde{F}$ is an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideal of S.

4.4 THEOREM

Let us define an interval-valued fuzzy subset $\tilde{F}$ of S by

$$\tilde{F}(x) = \begin{cases} \tilde{1}, & \text{if } x \in A, \\ \tilde{\delta}, & \text{if } x \notin A. \end{cases}$$

where A is a non-empty subset of S. If $\tilde{F}$ is an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideal of S, then A is an interior-ideal of S.

Proof. Let $\tilde{F}$ be an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideal of S. If $a, b \in A$, then $\tilde{F}(a) = \tilde{1} = \tilde{F}(b)$. By (m) of Theorem (4.3),

$$\begin{aligned} \mathrm{r}\max\{\tilde{F}(ab), \tilde{\delta}\} &\ge \mathrm{r}\min\{\tilde{F}(a), \tilde{F}(b)\} \\ &= \mathrm{r}\min\{\tilde{1}, \tilde{1}\} \\ &= \tilde{1}. \end{aligned}$$

This shows $\tilde{F}(ab) = \tilde{1}$ and hence $ab \in A$.

If $x, a, y \in S$ such that $a \in A$, then $\tilde{F}(a) = \tilde{1}$ and by (n) of Theorem (4.3)

$$\begin{aligned} \text{r max}\{\tilde{F}(xay), \tilde{\delta}\} &\geq \tilde{F}(a) \\ &= \tilde{1} \end{aligned}$$

It follows that $\tilde{F}(xay) = \tilde{1}$, therefore $xay \in A$.

Finally, for $a, b \in S$ if $a \leq b \in A$, then $\tilde{F}(b) = \tilde{1}$ and by (l) of Theorem (4.3),

$$\begin{aligned} \text{r max}\{\tilde{F}(x), \tilde{\delta}\} &\geq \tilde{F}(y) \\ &= \tilde{1}. \end{aligned}$$

This implies that $\tilde{F}(a) = \tilde{1}$ and thus $a \in A$. Consequently A is an interior ideal of S.

4.5 DEFINITION

An interval-valued fuzzy subset $\tilde{F}$ of S is called interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy left (resp. right) ideal of S, if the following hold for all $x, y \in S$, $\tilde{t} \in D(\tilde{\delta}, 1]$:

(o) . $(\forall\ x \leq y)\ (x_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F} \Rightarrow y_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F})$,

(p) . $(xy)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F} \Rightarrow x_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$ (resp. $y_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$).

4.6 THEOREM

The following are equivalent for any interval-valued fuzzy subset $\tilde{F}$ of S.

(1). $\tilde{F}$ is an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy left (resp. right) ideal of S.

(2). For all $x, y \in S$,

(2.1). $(\forall\ x \leq y)\ (\text{r max}\{\tilde{F}(x), \tilde{\delta}\} \geq \tilde{F}(y))$,

(2.2). $\text{r max}\{\tilde{F}(xy), \tilde{\delta}\} \geq \tilde{F}(y)$ (resp. $\tilde{F}(x)$).

Proof. $(1) \Rightarrow (2)$. If $a, b \in S$ with $a \leq b$ such that $\text{r max}\{\tilde{F}(x), \tilde{\delta}\} < \tilde{t} \leq \tilde{F}(y)$ for some $\tilde{t} \in D(\tilde{\delta}, 1]$, then $a_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ and $b_{\tilde{t}} \in_{\tilde{\gamma}} \wedge q_{\tilde{\delta}} \tilde{F}$. This contradicts Condition (o) of Definition (4.5) and hence we accept that (2.1) is valid for all $x, y \in S$ with $x \leq y$.

Next, suppose that $\text{r max}\{\tilde{F}(ab), \tilde{\delta}\} < \tilde{F}(b)$ for some $a, b \in S$. Then there exists $\tilde{t} \in D(\tilde{\delta}, 1]$ such that $\text{r max}\{\tilde{F}(ab), \tilde{\delta}\} < \tilde{t} \leq \tilde{F}(b)$, in which it follows that $(ab)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ but $b_{\tilde{t}} \in_{\tilde{\gamma}} \wedge q_{\tilde{\delta}} \tilde{F}$, a contradiction and hence (2.2) is valid for all $x, y \in S$.

$(2) \Rightarrow (1)$. If $a, b \in S$ with $a \leq b$ and $a_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$, then $\tilde{F}(a) < \tilde{t}$ and by (2.1),

$$\begin{aligned} \tilde{F}(b) &\leq \text{r max}\{\tilde{F}(a), \tilde{\delta}\} \\ &< \text{r max}\{\tilde{t}, \tilde{\delta}\} \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \geq \tilde{\delta}, \\ \tilde{\delta}, & \text{if } \tilde{t} < \tilde{\delta}. \end{cases} \end{aligned}$$

This implies, $b_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$.

Finally, if $(ab)_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}}\ \tilde{F}$ for some $a, b \in S$, then $\tilde{F}(ab) < \tilde{t}$ and by (2.2),

$$\begin{aligned} \tilde{F}(b) &\leq \text{r max}\{\tilde{F}(ab), \tilde{\delta}\} \\ &< \text{r max}\{\tilde{t}, \tilde{\delta}\} \\ &= \begin{cases} \tilde{t}, & \text{if } \tilde{t} \geq \tilde{\delta}, \\ \tilde{\delta}, & \text{if } \tilde{t} < \tilde{\delta}. \end{cases} \end{aligned}$$

It follows that $b_{\tilde{t}}\ \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}} \tilde{F}$. Consequently, $\tilde{F}$ is an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy left ideal of S. Similarly, we can prove that $\tilde{F}$ is an interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy right ideal of S.

CONCLUDING REMARKS

The idea of using intervals instead of single numbers play an essential part in the contemporary mathematics and several other applied fields of sciences like system control theory, robotics, computer engineering and automata theory. The concept of interval-valued fuzzy set gained the attentions of researchers around the world. They investigated several characterisation of interval-valued fuzzy sets and successfully applied in aforementioned fields which can be seen in terms of research articles in highly reputed journals and well known conferences. In this regard, we determined a new generalization of interval-valued fuzzy interior ideals and interval-valued fuzzy left (right) ideals by introducing interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy interior ideals, interval-valued $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$-fuzzy left (right) ideals, interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy interior ideals and interval-valued $(\overline{\in}_{\tilde{\gamma}}, \overline{\in}_{\tilde{\gamma}} \vee \overline{q}_{\tilde{\delta}})$-fuzzy left (right) ideals of ordered semigroups. Further, examples are also constructed for the support of these new concepts. In addition, several classes of ordered semigroups such as regular ordered semigroups and semisimple ordered semigroups are also characterised by the properties of these new notions. Lastly, the link between interval-valued fuzzy interior ideals and interval-valued fuzzy interior ideals of type $(\in_{\tilde{\gamma}}, \in_{\tilde{\gamma}} \vee q_{\tilde{\delta}})$ is constructed. These new investigations will fill the gap present in those applied fields which are using interval-valued fuzzy sets.

FUTURE WORK

These new ideas presented in this paper can also be applied in other algebraic structures like, Ring theory, Semigroups and Hemirings.

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